

# On the growth of ridge functions non-vanishing in an angular domain

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## Abstract

For an entire ridge function of finite order  $\rho$  which is non-vanishing in the angle  $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$ ,  $0 < \alpha \leq \pi/2$ , the sharp estimate of  $\rho$  in terms of  $\alpha$  is obtained. Analogous result is obtained for ridge functions analytic in the upper half-plane.

**Key words:** Characteristic function, Fourier transform, ridge function

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An entire function  $f$ ,  $f(0) = 1$ , is called a ridge function if it satisfies the inequality

$$|f(z)| \leq f(i\operatorname{Im} z). \quad (1)$$

If  $f$  is analytic in the upper half-plane  $\mathbf{C}_+$  and satisfies (1) there, then we call it an analytic in  $\mathbf{C}_+$  ridge function. In particular, all entire or analytic in  $\mathbf{C}_+$  characteristic functions of probability distributions are ridge functions.

The well-known Marcinkiewicz theorem [1] states that, if entire ridge function of finite order  $\rho$  has a few zeros in some sense, then  $\rho \leq 2$ . This theorem has been strengthened and generalized in many directions (see the bibliography in [2]). In particular, I. P. Kamynin [3] proved that, if an analytic in  $\mathbf{C}_+$  ridge function of a finite order  $\rho$  has no zeros at all, then  $\rho \leq 3$ . The examples of the entire characteristic function of the Gauss distribution  $f(z) = \exp(-\gamma z^2 + i\beta z)$ , ( $\gamma > 0$ ,  $\beta \in \mathbf{R}$ ), and the analytic in  $\mathbf{C}_+$  characteristic function  $f(z) = (1 - iz)^{-1} \exp(iz^3 - 3z^2)$  (constructed by I. P. Kamynin) show that both estimates are the best possible. In [4]

the question about estimate of the order of an entire or analytic in  $\mathbf{C}_+$  ridge function non-vanishing in some angle is considered. The following two theorems are proved.

**Theorem A** ([4]). If an analytic in  $\mathbf{C}_+$  ridge function of finite order  $\rho$  does not vanish in the angle  $\{z : |\arg z - \pi/2| < \alpha\}$ ,  $0 < \alpha \leq \pi/2$ , then  $\rho < \max(4; \pi/\alpha)$ .

**Theorem B** ([4]). If an entire ridge function of finite order  $\rho$  does not vanish in the angle  $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$ ,  $0 < \alpha \leq \pi/2$ , then

$$\rho \begin{cases} \leq 2, & \pi/4 \leq \alpha \leq \pi/2; \\ < \pi/\alpha, & 0 < \alpha < \pi/4. \end{cases} \quad (2)$$

We show that the estimates for  $\rho$  given in Theorem A and Theorem B can be improved and obtain the best possible estimates.

Denote by  $\gamma(\alpha)$ ,  $0 < \alpha \leq \pi/6$ , the solution of the equation

$$\cos^\gamma(\alpha + \pi/\gamma) = -\cos(\gamma\alpha) \quad (3)$$

which belongs to the interval  $(\pi/(2\alpha), \pi/\alpha)$ . It is easy to see that the function  $\gamma(\alpha)$  decreases and the following asymptotic equality holds

$$\gamma(\alpha) = \frac{\pi}{\alpha} - 2\sqrt{\frac{\pi}{\alpha}}(1 + o(1)), \quad \alpha \rightarrow 0. \quad (4)$$

**Theorem 1.** Let  $f$  be an analytic in  $\mathbf{C}_+$  ridge function of finite order  $\rho$  non-vanishing in the angle  $\{z : |\arg z - \pi/2| < \alpha\}$ ,  $0 < \alpha \leq \pi/2$ . Then

$$\rho \leq \begin{cases} \gamma(\alpha), & 0 < \alpha \leq \pi/6; \\ 3 = \gamma(\pi/6), & \pi/6 < \alpha \leq \pi/2. \end{cases} \quad (5)$$

**Theorem 2.** Let  $f$  be an entire ridge function of finite order  $\rho$  non-vanishing in the angle  $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$ ,  $0 < \alpha \leq \pi/2$ . Then

$$\rho \leq \begin{cases} \gamma(\alpha), & 0 < \alpha \leq \pi/6; \\ \pi/(2\alpha), & \pi/6 < \alpha \leq \pi/4; \\ 2, & \pi/4 < \alpha \leq \pi/2. \end{cases} \quad (6)$$

Remark. In [5], in particular, the following theorem has been proved.

**Theorem.** If an analytic in  $\mathbf{C}_+$  ridge function  $f$  does not vanish in the angle  $\{z : |\arg z - \pi/2| < \beta\}$  for some  $\beta \in (0; \pi/2]$  and

$$\lim_{y \rightarrow +\infty} \inf_y y^{-1} \log^+ \log^+ |f(iy)| = 0,$$

then the function  $f$  is of finite order.

Using this theorem, it is possible to weaken the condition of finiteness of the order in theorems 1 and 2.

Proof of Theorem 1. It is sufficient to prove the theorem for  $\alpha \in (0; \pi/6)$ . Assume that Theorem 1 is not valid. Let  $f$  be an analytic in  $\mathbf{C}_+$  ridge function which does not vanish in the angle  $\{z : |\arg z - \pi/2| < \alpha\}$ ,  $0 < \alpha < \pi/6$ , and has the finite order  $\rho > \gamma(\alpha) \geq \gamma(\pi/6) = 3$ . Let  $a_k = r_k e^{i\varphi_k}$  be zeros of  $f(iz)$ . We shall use the following notations:

$$u(z) = \log |f(iz)|; \quad (7)$$

$$v_R(z) = \operatorname{Im}(e^{i\rho\alpha} z^{-\rho} + z^\rho e^{-i\rho\alpha} R^{-2\rho}) = (|z|^{-\rho} - |z|^\rho R^{-2\rho}) \sin \rho(\alpha - \arg z); \quad (8)$$

$$\Pi_R = \{z : 1 < |z| < R, 0 < \arg z < \alpha + \pi/\rho\} \quad (R > 1); \quad (9)$$

$$\beta = \alpha + \pi/\rho \quad (0 < \beta < \pi/2). \quad (10)$$

Let us apply the Green formula in the domain  $\Pi_R$  to the pair  $(u(z), v_R(z))$ . Since  $v_R(z)$  is harmonic in  $\Pi_R$  for every  $R > 1$  and, moreover, from (1) follows  $\partial u / \partial y|_{(x,0)} = 0$ ,  $1 \leq x \leq R$ , we obtain

$$\begin{aligned} & \int_1^R \{(-\cos \rho\alpha)u(x) - u(xe^{i\beta})\} (x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx = \\ & 2R^{-\rho} \int_0^\beta u(Re^{i\theta}) \sin \rho(\alpha - \theta) d\theta + \\ & 2\pi\rho^{-1} \sum_{a_k \in \Pi_R} (r_k^{-\rho} - r_k^\rho R^{-2\rho}) \sin \rho(\alpha - \varphi_k) + C_1 + C_2 R^{-2\rho}, \end{aligned} \quad (11)$$

where  $C_1$  and  $C_2$  are constants not depending on  $R$ .

Denote

$$A(R) = \int_1^R \{(-\cos \rho\alpha)u(x) - u(xe^{i\beta})\} (x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx; \quad (12)$$

$$B(R) = 2R^{-\rho} \int_0^\beta u(Re^{i\theta}) \sin \rho(\alpha - \theta) d\theta; \quad (13)$$

$$S(R) = 2\pi\rho^{-1} \sum_{a_k \in \Pi_R} (r_k^{-\rho} - r_k^\rho R^{-2\rho}) \sin \rho(\alpha - \varphi_k). \quad (14)$$

Using these notations we can rewrite formula (11) in such a way

$$A(R) = B(R) + S(R) + C_1 + C_2 R^{-2\rho}. \quad (15)$$

Subtracting from (15) the formula obtained by replacing  $R$  by  $r$  in (15),  $1 < r < R < \infty$ , we obtain

$$A(R) - A(r) = B(R) - B(r) + S(R) - S(r) + C_2(R^{-2\rho} - r^{-2\rho}). \quad (16)$$

Now let us estimate from below the left-hand side of (16). Since  $f$  is a ridge function,  $u(x)$  is convex on  $R$  ([1]). Therefore without loss of generality we may assume that  $u(x)$  is positive and monotonically increases in  $x$  when  $x > 0$ . Further we shall denote by  $K$  positive constants not depending on  $r$ ,  $R$  not necessary equal. We have

$$\begin{aligned} A(R) - A(r) &= \int_1^R (u(x) - u(xe^{i\beta})) (x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx - \\ &\quad \int_1^r (u(x) - u(xe^{i\beta})) (x^{-\rho-1} - x^{\rho-1} r^{-2\rho}) dx - \\ &\quad (1 + \cos \rho\alpha) \int_1^R u(x) (x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx + \\ &\quad (1 + \cos \rho\alpha) \int_1^r u(x) (x^{-\rho-1} - x^{\rho-1} r^{-2\rho}) dx \geq \\ &\quad \int_r^R (u(x) - u(xe^{i\beta})) (x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx - \\ &\quad (1 + \cos \rho\alpha) \int_r^R u(x) x^{-\rho-1} dx - (1 + \cos \rho\alpha) \int_1^r u(x) x^{\rho-1} r^{-2\rho} dx \geq \\ &\quad \int_r^R (u(x) - u(x \cos \beta)) (x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx - \\ &\quad (1 + \cos \rho\alpha) \int_r^R u(x) x^{-\rho-1} dx - (1 + \cos \rho\alpha) \int_1^r u(x) x^{\rho-1} r^{-2\rho} dx \geq \\ &\quad \int_r^R \{(-\cos \rho\alpha) u(x) - u(x \cos \beta)\} x^{-\rho-1} dx - \\ &\quad \int_r^R u(x) x^{\rho-1} R^{-2\rho} dx - (1 + \cos \rho\alpha) \int_1^r u(x) x^{\rho-1} r^{-2\rho} dx \geq \end{aligned}$$

$$\begin{aligned}
& (-\cos \rho \alpha) \int_r^R u(x) x^{-\rho-1} dx - \cos^\rho \beta \int_{r \cos \beta}^{R \cos \beta} u(x) x^{-\rho-1} dx - \\
& Ku(R)R^{-\rho} - Ku(r)r^{-\rho}(-\cos \rho \alpha - \cos^\rho \beta) \int_r^R u(x) x^{-\rho-1} dx - \\
& \cos^\rho \beta \int_{r \cos \beta}^r u(x) x^{-\rho-1} dx - Ku(R)R^{-\rho} - Ku(r)r^{-\rho} \geq \\
& (-\cos \rho \alpha - \cos^\rho \beta) \int_r^R u(x) x^{-\rho-1} dx - Ku(R)R^{-\rho} - Ku(r)r^{-\rho}. \quad (17)
\end{aligned}$$

The function  $h_\alpha(\rho) = -\cos \rho \alpha - \cos^\rho \beta = -\cos \rho \alpha - \cos^\rho(\alpha + \pi/\rho)$  has the unique root  $\gamma(\alpha)$  on the interval  $[\pi/(2\alpha); \pi/\alpha]$ , and  $h_\alpha(\pi/\alpha) > 0$ . Therefore using our assumption  $\rho > \gamma(\alpha)$  we have  $\varepsilon = h_\alpha(\rho) > 0$ . Substituting estimate (17) into (16) and dividing by  $\varepsilon > 0$  we obtain

$$\begin{aligned}
\int_r^R u(x) x^{-\rho-1} dx \leq K(B(R) - B(r)) + K(S(R) - S(r)) + \\
Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \quad (18)
\end{aligned}$$

(we have applied positivity and monotonic increasing of  $u(x)$ ).

Let us estimate the right-hand side of (18) from above.

**1.** To estimate  $B(R)$  we use the following Lemma.

**Lemma 1.** ([4])

$$|B(R)| \leq Ku(R)R^{-\rho}. \quad (19)$$

Proof of Lemma 1. We use the Carleman formula for function  $f(z)$  ([8, p.224]):

$$\begin{aligned}
\frac{1}{\pi R} \int_0^\pi \log |f(Re^{i\theta})| \sin \theta d\theta + \frac{1}{2} \int_1^R (t^{-2} - R^{-2}) \log |f(t)f(-t)| dt = \\
\sum_{1 < r_k < R} (r_k^{-1} - r_k R^{-2}) \cos \varphi_k + b_1 + b_2 R^{-2}, \quad (20)
\end{aligned}$$

where  $1 \leq R < \infty$ ;  $b_1, b_2$  do not depend on  $R$ . Since  $f(z)$  is a ridge function,  $f(0) = 1$ , we have  $|f(t)| \leq 1$ ,  $t \in \mathbf{R}$ . Hence (20) implies

$$\frac{1}{\pi R} \int_0^\pi \log |f(Re^{i\theta})| \sin \theta d\theta \geq b_1 + b_2 R^{-2}. \quad (21)$$

Using the fact that  $0 < \beta < \pi/2$  and (21), we obtain

$$|B(R)| \leq KR^{-\rho} \int_{\pi/2}^{\pi/2+\beta} |\log |f(Re^{i\theta})|| d\theta \leq$$

$$\begin{aligned}
KR^{-\rho} \int_{\pi/2}^{\pi/2+\beta} (\log^+ |f(Re^{i\theta})| + \log^- |f(Re^{i\theta})|) \sin \theta d\theta &\leq \\
KR^{-\rho} u(R) + KR^{-\rho} \int_0^\pi \log^- |f(Re^{i\theta})| \sin \theta d\theta &\leq \\
KR^{-\rho} (u(R) + \pi b_1 R + \pi b_2 R^{-1}). &\quad (22)
\end{aligned}$$

Since  $f$  is an unbounded in  $\mathbf{C}_+$  analytic ridge function we have by [1]  $R = O(\log^+ f(iR))$ ,  $R \rightarrow +\infty$ . Therefore from (22) we obtain the statement of Lemma 1. ■

Using the statement of Lemma 1 we obtain

$$B(R) - B(r) \leq Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \quad (23)$$

**2.** To estimate  $S(R) - S(r)$  we shall use the fact that function  $f(iz)$  does not vanish in the angle  $\{z : 0 < \arg z < \alpha\}$ . We have

$$\begin{aligned}
S(R) - S(r) &= 2\pi\rho^{-1} \left( \sum_{\substack{1 < r_k < R \\ \alpha < \varphi_k < \beta}} (r_k^{-\rho} - r_k^\rho R^{-2\rho}) \sin \rho(\alpha - \varphi_k) - \right. \\
&\quad \left. \sum_{\substack{1 < r_k < r \\ \alpha < \varphi_k < \beta}} (r_k^{-\rho} - r_k^\rho r^{-2\rho}) \sin \rho(\alpha - \varphi_k) \right) = \\
&= 2\pi\rho^{-1} \left( \sum_{\substack{1 < r_k < r \\ \alpha < \varphi_k < \beta}} r_k^\rho (r^{-2\rho} - R^{-2\rho}) \sin \rho(\alpha - \varphi_k) + \right. \\
&\quad \left. \sum_{\substack{r < r_k < R \\ \alpha < \varphi_k < \beta}} (r_k^{-\rho} - r_k^\rho R^{-2\rho}) \sin \rho(\alpha - \varphi_k) \right) \leq 0. \quad (24)
\end{aligned}$$

Substituting (23) and (24) into (18), we obtain

$$\int_r^R u(x) x^{-\rho-1} dx \leq Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \quad (25)$$

To complete the proof we shall use the following elementary lemma:

**Lemma 2** ([4]). Let  $w(x) \geq 0$  be continuously differentiable non-decreasing function on  $[1; \infty)$  and for all  $r$  and  $R$ ,  $r < R$ , the inequality

$$w(R) - w(r) \leq KRw'(R) + Krw'(r). \quad (26)$$

holds. Then for all sufficiently large  $x$  we have:

- 1) if  $w(\infty) = \infty$ , then  $w(x) \geq Kx^\delta$ ;
- 2) if  $w(\infty) < \infty$ , then  $w(\infty) - w(x) \leq Kx^{-\delta}$ ,

where  $\delta$  is a positive number.

Proof of Lemma 2.

**1.** Let  $w(\infty) = \infty$ . Then by (26) we have  $Rw'(R) \rightarrow \infty$ ,  $R \rightarrow \infty$ . Substituting  $r = 1$  into (26) and using the last statement, we have

$$w(R) \leq KRw'(R) \quad (27)$$

(with a larger  $K$ ).

Hence

$$\frac{w'(R)}{w(R)} \geq \frac{1}{KR}. \quad (28)$$

Integrating the last inequality from 1 to  $r$ , we obtain statement 1) of Lemma 2.

**2.** Let  $w(\infty) < \infty$ . Then there exists a sequence  $R_k \rightarrow \infty$  such that  $R_k w'(R_k) \rightarrow 0$ ,  $k \rightarrow \infty$ . Making  $R \rightarrow \infty$  along the sequence  $\{R_k\}_{k=1}^\infty$  we have

$$w(\infty) - w(r) \leq Krw'(r), \quad (29)$$

hence

$$\frac{w'(r)}{w(\infty) - w(r)} \geq \frac{1}{Kr} \quad (30)$$

Integrating last inequality from 1 to  $r$ , we obtain

$$-\log \frac{w(\infty) - w(r)}{w(\infty) - w(1)} \geq \log r^{1/K}, \quad (31)$$

that implies statement 2) of Lemma 2. ■

Using Lemma 2 with  $w(x) = \int_1^x u(t)t^{-\rho-1}dt$ , we obtain contradiction to the assumption that  $\rho$  is the order of the function  $f$ .

Theorem 1 is proved. ■ ■

Proof of the Theorem 2. Since  $f$  is an analytic in  $C_+$  ridge function we have  $\rho \leq \gamma(\alpha)$  for  $0 < \alpha \leq \pi/6$ . Therefore it is sufficient to prove Theorem 2 for  $\pi/6 < \alpha < \pi/4$ ,  $2 < \rho \leq 3$ . Without loss of generality we can assume that  $f$  is an even function (we can consider the function  $f(z)f(-z)$  instead of  $f(z)$ ). Suppose the contrary that there exists an even entire ridge function  $f$  non-vanishing in the angle  $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$

and having the finite order  $\rho > \pi/(2\alpha)$ . Let us denote by  $a_k = r_k e^{i\varphi_k}$  zeros of function  $f(iz)$  lying in the half-plane  $\{z : \operatorname{Re} z > 0\}$ . We use the notations  $u(z)$  and  $v_R(z)$  introduced in the proof of the Theorem 1. Denote  $C_R = \{z : 1 < |z| < R, 0 < \arg z < \pi/2\}$ ,  $R > 1$ . Let us use the Green formula in  $C_R$  for functions  $u$  and  $v_R$ . Since  $v_R$  is harmonic in  $C_R$ , and  $f(z)$  is even and satisfies (1), we obtain  $\partial u(x, 0)/\partial y = 0$ ,  $\partial u(0, y)/\partial x = 0$ . Hence

$$\begin{aligned} \int_1^R \{(-\cos \rho\alpha)u(x) - (-\cos \rho(\alpha - \pi/2))u(ix)\}(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx = \\ 2R^{-\rho} \int_0^{\pi/2} u(Re^{i\theta}) \sin \rho(\alpha - \theta) d\theta + \\ 2\pi\rho^{-1} \sum_{a_k \in C_R} (r_k^{-\rho} - r_k^{\rho}R^{-2\rho}) \sin \rho(\alpha - \varphi_k) + C_1 + C_2R^{-2\rho}, \end{aligned} \quad (32)$$

where  $C_1$  and  $C_2$  are positive constants. By our assumption  $(-\cos \rho(\alpha - \pi/2)) > 0$  holds. Since  $f$ ,  $f(0) = 1$  is an even entire ridge function, the function  $u(x)$  is positive and monotonically increases in  $x$ ,  $x > 0$ . As in the proof of Theorem 1 we denote

$$A(R) = \int_1^R \{(-\cos \rho\alpha)u(x) - (-\cos \rho(\alpha - \pi/2))u(ix)\} \times (x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx; \quad (33)$$

$$B(R) = 2R^{-\rho} \int_0^{\pi/2} u(Re^{i\theta}) \sin \rho(\alpha - \theta) d\theta; \quad (34)$$

$$S(R) = 2\pi\rho^{-1} \sum_{a_k \in C_R} (r_k^{-\rho} - r_k^{\rho}R^{-2\rho}) \sin \rho(\alpha - \varphi_k). \quad (35)$$

Subtracting from (32) the formula obtained from (32) by changing  $R$  by  $r$ ,  $1 < r < R < \infty$ , we have

$$A(R) - A(r) = B(R) - B(r) + S(R) - S(r) + C_2(R^{-2\rho} - r^{-2\rho}). \quad (36)$$

Let us estimate the left-hand part of (36) from below

$$\begin{aligned} A(R) - A(r) = \int_1^R (-\cos \rho\alpha)u(x)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx - \\ \int_1^r (-\cos \rho\alpha)u(x)(x^{-\rho-1} - x^{\rho-1}r^{-2\rho})dx - \\ (-\cos \rho(\alpha - \pi/2)) \int_1^R u(ix)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx + \end{aligned}$$



$$\begin{aligned}
& (-\cos \rho(\alpha - \pi/2)) \int_1^r u(ix)(x^{-\rho-1} - x^{\rho-1}r^{-2\rho})dx \geq \\
& x(-\cos \rho\alpha) \int_1^R u(x)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx - \\
& (-\cos \rho\alpha) \int_1^r u(x)(x^{-\rho-1} - x^{\rho-1}r^{-2\rho})dx \geq \\
& (-\cos \rho\alpha) \int_r^R u(x)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx \geq \\
& (-\cos \rho\alpha) \int_r^R u(x)x^{-\rho-1}dx - Ku(R)R^{-\rho}. \tag{37}
\end{aligned}$$

The upper estimate of  $B(R) - B(r)$  we obtain in the same way as in the proof of Theorem 1. We have

$$B(R) - B(r) \leq Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \tag{38}$$

Since function  $f$  has no zeros in the angle  $\{z : |\arg z - \pi/2| < \alpha\}$

$$S(R) - S(r) \leq 0 \tag{39}$$

holds.

Substituting (37), (38), and (39) into (36), we obtain

$$\int_r^R u(x)x^{-\rho-1}dx \leq Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \tag{40}$$

As in the proof of Theorem 1 we see that (40) contradicts to the fact that  $\rho$  is the order of function  $f$ .

Theorem 2 is proved. ■ ■

The next statement shows the sharpness of the estimates of  $\rho$  in Theorems 1 and 2.

**Theorem 3. 1.** For each  $\alpha$ ,  $0 < \alpha \leq \pi/2$ , there exists an entire characteristic function  $f$  nonvanishing in the angle  $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$  and having the order

$$\rho = \begin{cases} \gamma(\alpha), & 0 < \alpha \leq \pi/6; \\ \pi/(2\alpha), & \pi/6 < \alpha \leq \pi/4; \\ 2, & \pi/4 < \alpha \leq \pi/2. \end{cases} \tag{41}$$

**2.** For each  $\alpha$ ,  $0 < \alpha \leq \pi/2$ , there exists an analytic in  $\mathbf{C}_+$  characteristic function  $f$  non-vanishing in the angle  $\{z : |\arg z - \pi/2| < \alpha\}$  and having the order

$$\rho = \begin{cases} \gamma(\alpha), & 0 < \alpha \leq \pi/6; \\ 3, & \pi/6 < \alpha \leq \pi/2. \end{cases} \quad (42)$$

Proof of Theorem 3. **1.** For  $\alpha \in [\pi/4; \pi/2]$  we can take the Gauss characteristic function  $f(z) = \exp\{-\gamma z^2 + i\beta z\}$  ( $\gamma > 0$ ;  $\beta \in \mathbf{R}$ ) as an example. Consider  $\alpha \in (0; \pi/4)$ . For constructing an example in this case we shall use a result of [6]. Let  $h(\theta)$  be a  $2\pi$ -periodic function on  $\mathbf{R}$ , let  $\rho$  be a number greater than 2. Suppose that the following conditions are satisfied:

- 1)  $h(\theta)$  is a  $\rho$ -trigonometrically convex function;
- 2)  $\exists \delta > 0, A > 0; h(\theta) = A \cos \rho(\pi/2 - \theta)$  for  $|\pi/2 - \theta| < \delta$ ;
- 3)  $h(\pi/2 + \theta) = h(\pi/2 - \theta)$ , for  $\theta \in [0; \pi/2]$ ;
- 4)  $h(\pi/2 + \theta) \leq h(\pi/2) \cos^\rho(\pi/2 - \theta)$ , for  $\theta \in [0; \pi/2]$ .

**Theorem [6].** There exists an entire characteristic function  $f$  of order  $\rho$  having completely regular growth (in sense Levin-Pfluger) with the indicator  $h(\theta)$  and non-vanishing inside the angles where  $h(\theta)$  is  $\rho$ -trigonometric.

We shall construct function  $h(\theta)$  satisfying conditions 1)–4) and such that  $h(\theta)$  is  $\rho$ -trigonometric for  $\theta \in [\pi/2 - \alpha; \pi/2 + \alpha]$ . Since  $h(\theta)$  will be an even function satisfying 3), it is sufficient to construct  $h(\theta)$  when  $\theta \in [\pi/2; \pi]$ .

**a).** Consider  $\alpha \in [0; \pi/6]$ ,  $\rho = \gamma(\alpha) \geq 3$ . Define  $h(\theta)$  by the formula

$$h(\theta) = \begin{cases} \cos \rho(\pi/2 - \theta), & \theta \in [\pi/2; \pi/2 + \alpha]; \\ -\cos^{\rho-1}(\alpha + \pi/\rho) \times \\ \cos \rho(\theta + \alpha/\rho + \pi/\rho^2 - \alpha - \pi/2), & \theta \in [\pi/2 + \alpha; \pi/2 + \alpha + \pi/\rho]; \\ \cos^\rho(\pi/2 - \theta), & \theta \in [\pi/2 + \alpha + \pi/\rho; \pi]. \end{cases} \quad (43)$$

It is easy to verify that  $h(\theta)$  is a  $\rho$ -trigonometrically convex function (we use the fact that  $\rho = \gamma(\alpha)$  satisfies equation  $\cos^\rho(\alpha + \pi/\rho) = -\cos \rho \alpha$ ) and  $h(\theta)$  satisfies 1)–4). Therefore Theorem of [6] cited above yields that there exists an entire characteristic function  $f$  of order  $\rho$  and of completely regular growth having indicator  $h(\theta)$ . Since  $h(\theta)$  is  $\rho$ -trigonometric for  $\theta \in [-\pi/2 - \alpha; -\pi/2 + \alpha] \cup [\pi/2 - \alpha; \pi/2 + \alpha]$ , the function  $f$  does not vanish in the angle  $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$ .

2. Consider  $\alpha \in [\pi/6; \pi/4]$ ,  $\rho = \pi/(2\alpha)$ . Define

$$h(\theta) = \begin{cases} \cos \rho(\pi/2 - \theta), & \theta \in [\pi/2; \pi/2 + \alpha]; \\ 0, & \theta \in [\pi/2 + \alpha; \pi]. \end{cases} \quad (44)$$

The function  $h(\theta)$  is  $\rho$ -trigonometrically convex and satisfies 1)–4). Therefore the above Theorem of [6] yields that there exists an entire characteristic function  $f$  of order  $\rho$  and of completely regular growth having indicator  $h(\theta)$ . Since  $h(\theta)$  is  $\rho$ -trigonometric for  $\theta \in [-\pi/2 - \alpha; -\pi/2 + \alpha] \cup [\pi/2 - \alpha; \pi/2 + \alpha]$ , the function  $f$  does not vanish in the angle  $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$ .

2. When  $\alpha \in (0; \pi/6]$  we can take as example the entire function constructed in a). When  $\alpha \in [\pi/6; \pi/2]$  we can take  $f(z) = (1 - iz)^{-1} \exp(iz^3 - 3z^2)$ .

Theorem 3 is proved. ■ ■

Remark. Using methods of the theory of the cluster sets of subharmonic functions developed by V. S. Azarin [9], A. E. Fryntov in [7] proved independently some more general statement than Theorem 2.

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